SYNODIC PERIODS AND ORBITAL ECCENTRICITY

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The effect of orbital eccentricity on the synodic period of a planet is examined at an undergraduate level. In the case of Mars, the effect is not dramatic but is certainly detectable in that the times between just a few successive oppositions can vary by values on the order of a month. This analysis could be appropriate as supplemental classroom material or as the source of a homework exercise.

A standard element of traditional astronomy education is to show how Copernicus and Kepler used measured synodic (s) periods of planets to determine their sidereal (T) periods, with the two being related by

$$T = \frac{s}{(s \mp 1)}, \qquad (1)$$

where the upper (lower) sign holds for a superior (inferior) planet and all periods are measured in Earth years. Recall that the sidereal period is the physically important one, being the time required for a planet to orbit the Sun with respect to a distant star as seen by an observer outside the Solar System; this is the period that appears in Kepler's Third Law. However, this cannot be observed directly by Earth-bound observers, as we also orbit the Sun. Rather, what we can measure is the synodic period, the time required for a planet to return to the same location in the sky relative to the Earth and Sun, typically a conjunction, opposition, or elongation. I will use the term 'alignment' in a generic sense to cover all these possibilities.

An informal survey of various lower-level undergraduate texts reveals that some derive Eq. (1) while others simply quote it or give only a qualitative description. Some apply the concept with a focus on lunar phases more so than planetary orbits, while in others the reverse is the case; my concern here is with planets. Whatever the level of treatment, however, the assumption is always that the orbits are *circular*, although some do remark that this is an approximation. An inquisitive student might then ask: "How would the eccentricity, even if modest, affect the synodic period? Also, since the speed of a planet in its orbit is always varying, would the time between successive alignments be truly periodic?"

This paper describes an analysis of this situation and a program developed to run corresponding numerical calculations. For simplicity, I do assume that Earth's orbit is circular, while that of the target planet has some eccentricity ε.

Fig. 1 illustrates an alignment between the Earth and a superior planet, whose major axis lies along the horizontal direction. At the moment of alignment, the apsidal angles φ_E of the Earth and φ_P of the planet are identical. Call this common angle φ_0, and the time at which this occurs to be t_0. The problem is to determine the angular position and time of the next alignment. Note that this does not require that the Earth and planet again align along the specific direction φ_0, only that they align along the same value of φ. All angles are measured in radians.

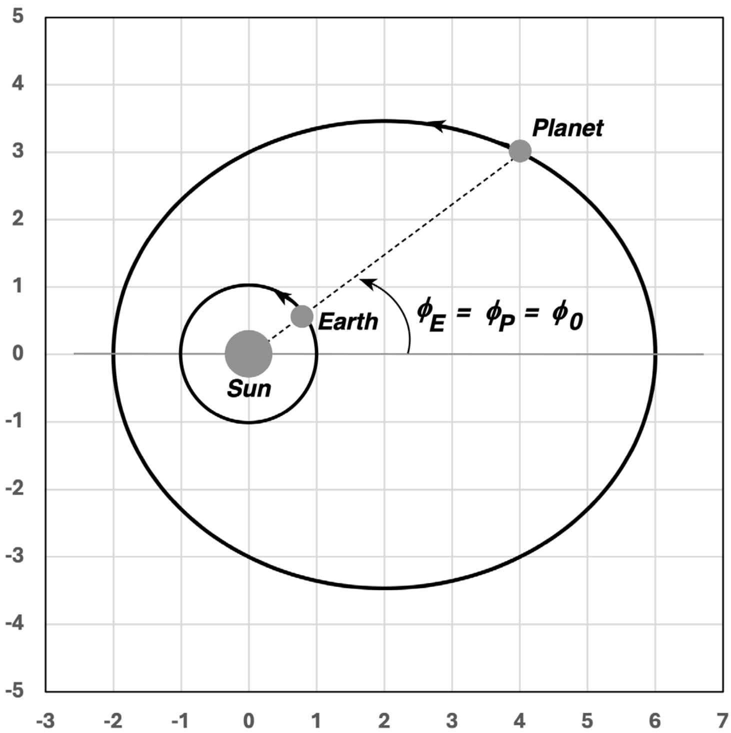

FIG. 1

Scale drawing of Earth with a superior planet in opposition at common apsidal angle $\varphi_E = \varphi_P = \varphi_0$. Scales are in AU, with the Sun at the origin. The planetary orbit has $a = 4$ AU and $\varepsilon = 0{\cdot}5$.

For the Earth in its circular orbit with sidereal period T_E (later to be set to 1 year), the orbital angular speed is constant at $2\pi/T_E$. At some later time t, Earth will be at apsidal angle φ_E given by

$$(\varphi_E - \varphi_0) = \left(\frac{2\pi}{T_E}\right)(t - t_0). \qquad (2)$$

Now, it is shown below that for the planet with sidereal period T_P, any later time t will correspond to apsidal position φ_P according as

$$(t - t_0) = \frac{(1-\varepsilon^2)^{3/2}\, T_P}{2\pi}[f(\varepsilon, \varphi_P) - f(\varepsilon, \varphi_0)], \qquad (3)$$

where the function $f(\varepsilon, \varphi)$ is determined in Eq. (8) below.

If Eq. (2) is solved for $(t - t_0)$ and the result substituted into Eq. (3), we have

$$(\varphi_E - \varphi_0) = (1-\varepsilon^2)^{3/2}\left(\frac{T_P}{T_E}\right)[f(\varepsilon, \varphi_P) - f(\varepsilon, \varphi_0)]. \qquad (4)$$

The condition for the next alignment is that in the case of a superior (inferior) planet, the elapsed apsidal angle for the Earth has moved ahead (fallen behind) that of the planet by 2π radians:

$$(\varphi_E - \varphi_P) = \pm 2\pi, \tag{5}$$

where the upper (lower) sign again applies for a superior (inferior) planet. Using this expression for φ_E in Eq. (4) gives

$$(\varphi_E - \varphi_0) \pm 2\pi - (1-\varepsilon^2)^{3/2}\left(\frac{T_P}{T_E}\right)[f(\varepsilon, \varphi_P) - f(\varepsilon, \varphi_0)] = 0. \tag{6}$$

This is an equation of constraint for the next alignment at apsidal angle φ_P. Once this position has been determined, the corresponding time can be determined from Eq. (2) or (3), and the position and time can then be treated as (φ_0, t_0) for determining the next alignment. The process can then be continued for as many synods as desired.

To determine the function $f(\varepsilon, \varphi)$, we appeal to the standard result that for an elliptical orbit of eccentricity ε and period T, the time to travel from apsidal angle φ_0 to angle φ is given by

$$(t - t_0) = \frac{(1-\varepsilon^2)^{3/2}\,T}{2\pi}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{(1-\varepsilon\cos\varphi)^2}. \tag{7}$$

Unfortunately, the exact closed-form solution of this integral involves computing the inverse-tangent of the product of a factor involving ε times the tangent of $\varphi/2$. Whenever a calculation involves an inverse-tangent, quadrant ambiguities come into play. In the present case this is compounded by the fact that φ accumulates to several multiples of 2π radians as subsequent alignments are sought. To avoid this complication, I treat the integral by a binomial expansion of the denominator to second order in ε, presuming that ε is not too great. (For the same reason, I avoid introducing Kepler's equation, which also involves a tangent.) The function f of Eq. (4) is the indefinite integral:

$$f(\varphi, \varepsilon) = \int\frac{d\varphi}{(1-\varepsilon\cos\varphi)^2} = \int(1 + 2\varepsilon\cos\varphi + 3\varepsilon^2\cos^2\varphi + \ldots)d\varphi.$$

$$= \varphi(1 + \tfrac{3}{2}\varepsilon^2) + 2\varepsilon\sin\varphi + \tfrac{3}{4}\varepsilon^2\sin(2\varphi) + \ldots. \tag{8}$$

To evaluate these calculations, I prepared a double-precision FORTRAN program into which the user enters the desired planetary sidereal period T_P in years, the eccentricity, and the apsidal angle φ_0 of an initial alignment. The program computes the aphelion and perihelion distances of the planet to ensure that no Earth-orbit crossings occur, and then determines apsidal angles and times for 100 subsequent alignments.

Preliminary calculations indicated that the time between successive synods does indeed vary somewhat around the textbook value that would be computed from Eq. (1), so the program takes a brute-force approach. Beginning at a trial value of φ_0 + 0·02 radians and going in steps of 0·02 radians, the constraint equation is evaluated until it changes sign; a bisection routine is then used to pin down the apsidal angle of the next alignment to a tolerance of 10^{-8} radians. The 'initial angle' φ_0 is then reset to the position of the alignment so determined, and the calculations reiterated to determine the next alignment. The program runs

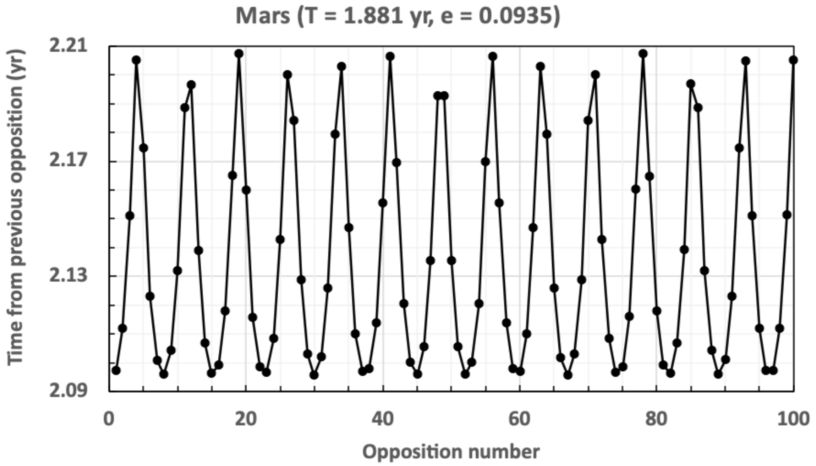

FIG. 2

Times between successive oppositions of Mars; $T = 1{\cdot}881$ yr, $\varepsilon = 0{\cdot}0935$, $\varphi_0 = 0$.

to about 250 lines including extensive comments and executes in a few seconds on a desktop computer.

Fig. 2 shows results obtained for Mars, for which a NASA website gives $T_P = 1{\cdot}881$ years and $\varepsilon = 0{\cdot}0935$.[1] Our inquisitive student is indeed correct. Synodic periods vary from about 2·09 to 2·21 years, a spread of some 44 days. The average over 100 synods, 2·1357 years (s.d. = 0·0380 years), is close to the value that would be computed from Eq. (1), 2.1351 years. Runs with increasing assumed eccentricity show a trend to increasing average, but also with increasing spread and with the nominal value always well within the spread.

The quasi-periodicity evident in Fig. 2 hints at a phenomenon known to ancient astronomers: that oppositions of Mars show a repeating pattern with respect to background stars in that nine nominal synodic periods of 2·1351 years corresponds to a little more than 19 years. Venus exhibits a similar effect, with five of its 583·9-day synodic periods spanning almost exactly eight years. This sort of effect is by no means guaranteed; the synodic period needs to be close to a rational-fraction number of years.

Synodic periods are now of largely historical interest, but it can be enjoyable to explore the nuances of what we learned in foundational classes. I would be happy to share the FORTRAN code with any interested reader.

Reference

(1) https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html